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A NOTE ON THE INVERSE SOURCE PROBLEM

by

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In an earlier paper, the authors derived a Fredholm integral equation of the first kind for the solution of the inverse source problem for acoustic waves. The eigenvalues of this equation were shown to converge rapidly to zero and also to include zero. Thus, the solution was shown to be non-unique and even the particular part of the solution of that equation was ill-conditioned. In this note it is shown how to obtain the non-trivial information of that integral equation in a well-conditioned manner.

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Abstract

In an earlier paper, the authors derived a Fredholm integral equation of the first kind for the solution of the inverse source problem for acoustic waves. The eigenvalues of this equation were shown to converge rapidly to zero and also to include zero. Thus, the solution was shown to be non-unique and even the particular part of the solution of that equation was ill-conditioned. In this note it is shown how to obtain the non-trivial information of that integral equation in a well-conditioned manner.

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In an earlier paper,[1] the authors developed analytical characterizations of the non-uniqueness of solutions of the inverse source problem in acoustics and electromagnetics. In this problem the field radiated by a source distribution is observed on a closed surface (e.g., a sphere) outside the region containing the source. The objective is to obtain information about the source.

It was further shown in Ref. 1, that the source distribution satisfies a Fredholm integral equation of the first kind with eigen-values which rapidly approached zero $0 (n^{-2n-3})$ in addition to the eigenvalue zero, itself. Thus, even the *particular solution* of this equation--for that part of the source which could be determined from the radiated field--is *ill-conditioned*; i.e., highly unstable to noise in the "higher" eigenfunctions.

The purpose of the note is to show that the information contained in that integral equation can, in fact, be obtained in a *well-conditioned* manner. The discussion here will be limited to the acoustic case; the extension to the electromagnetic case is straightforward.

It is assumed that $u(\underline{x}, \omega)$, $\underline{x} = (x_1, x_2, x_3)$, is a solution of the inhomogeneous Fourier (time) transformed wave equation,

$$(\nabla^2 + \omega^2 c^{-2})\ u(\underline{x}, \omega) = -f(\underline{x}, \omega), \tag{1}$$

subject to the radiation condition

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$$u(\underline{x}, \omega) \sim u_0(\hat{x}, \omega) \exp\{i\omega x/c\} /(4\pi x), \quad x \to \infty \quad . \tag{2}$$

Here, $\hat{x}$ denotes a unit vector in the direction of $\underline{x}$ and $x = |\underline{x}|$ is the magnitude of the vector $\underline{x}$.

The source function $f(\underline{x}, \omega)$ is assumed to be confined to the interior of a sphere of radius a. For observations outside of this sphere, the solution has the integral representation

$$u(\underline{x}, \omega) = i\omega c^{-1} \sum_{m=0}^{\infty} \sum_{n=-m}^{m} c_{mn} h_m^{(1)}(\omega x/c) Y_{mn}(\theta, \phi). \tag{3}$$

Here,

$$c_{mn} = \int_0^a f_{mn}(x, \omega) j_m(\omega x/c) x^2 dx \quad ; \tag{4}$$

j_m and $h_m^{(1)}$ are respectively, the spherical Bessel function and Hankel function of the first kind and Y_{mn} is the spherical harmonic of order mn. The functions $f_{mn}(x, \omega)$ are defined by

$$f_{mn}(x, \omega) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \, f(\underline{x}, \omega) Y_{mn}^*(\theta, \phi). \tag{5}$$

These functions are the coefficients of f in its spherical harmonic expansion

$$f(\underline{x}, \omega) = \sum_{n=0}^{\infty} \sum_{n=-m}^{m} f_{mn} (x, \omega) Y_{mn}(\theta, \phi). \qquad (6)$$

The representation (3) arises from the Green's function representation of the solution[2] and the spherical harmonic expansion of the Green's function in Ref. 2 on p. 742.

Since the spherical harmonics are a complete set of functions, knowledge of the coefficients f_{mn} constitutes knowledge of f itself. However, from (3, 4, 5) it is seen that the radiated field (i.e., u for x > a) is a function, not of f_{mn}'s, but of their *projections* c_{mn} on the spherical Bessel functions $\{j_m\}$. Since the set of functions $\{j_m Y_{mn}\}$ are not complete, this led in Ref. 1 to one analytical characterization of the non-uniqueness of the solution of the inverse source problem.

In the particular solution of the inverse source problem in terms of spherical harmonics, one could only hope to determine the coefficients c_{mn} and *not* the coefficients f_{mn}. It will now be shown how this can be done without resorting to an ill-conditioned integral equation.

To begin, the function $v(\underline{x}, \omega)$ is introduced, denoting any solution of the homogeneous reduced wave equation in the region D, on the boundary of which (∂D) the radiated field, $u(\underline{x}, \omega)$ is to be observed. Now Green's theorem is applied on the domain D to the quantity

$$v(\underline{x}, \omega)[\nabla^2 + \omega^2 c^{-2}]u(\underline{x}, \omega) - u(\underline{x}, \omega)[\nabla^2 + \omega^2 c^{-2}] v(\underline{x}, \omega).$$

The result is

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$$\int\limits_{x < a} v(\underline{x}, \omega) \ f(\underline{x}, \omega) \ dV = \int\limits_{\partial D} \left\{ u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right\} \ dA \ . \qquad (7)$$

For any function $v(\underline{x}, \omega)$ of the prescribed type, this is an integral equation for $f(\underline{x}, \omega)$. A particular choice suggested by the discussion above is any function in the class

$$\Psi_{mn} = j_m(\omega x/c)Y_{mn}(\theta,\phi), \ m = 1, \ 2, \ ..., \ |n| \leq m \ . \qquad (8)$$

With $v(\underline{x}, \omega) = \Psi_{mn}^*$, (7) becomes

$$c_{mn} = \int\limits_{\partial D} \left\{ u\frac{\partial \Psi^*}{\partial n} - \Psi^*\frac{\partial u}{\partial n} \right\} \ dA. \qquad (9)$$

Thus, the c_{mn}'s are given in terms of well-conditioned operations--multiplication by known functions and integration--on the observed data.

In Ref. 1, an equivalent characterization of non-uniqueness was given in terms of the space-time transform, $\tilde{f}(\underline{k}, \omega)$, of the source distribution. In particular, it was shown that the radiated field was given *totally* in terms of the value of $\tilde{f}(\underline{k}, \omega)$ on the hyper-cone where

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$$k = |\omega|/c \quad . \tag{10}$$

Conversely, from observations of the radiated field and its normal derivative one can only determine $\tilde{f}(\underline{k}, \omega)$ on this hyper-cone.

These function values can also be extracted from (7) in a well-conditioned manner. To do so, one need only choose v to be in the continuum of functions over unit vectors $\hat{k}$,

$$v(\underline{x}, \omega, \hat{k}) = \exp\{i\omega\hat{k} \cdot \underline{x}/c\} \quad . \tag{11}$$

Now (7) becomes

$$\tilde{f}(\underline{k}, \omega) = \int\limits_{\partial D} \left\{ i\underline{k} \cdot \hat{n} \, u - \frac{\partial u}{\partial n} \right\} \exp\{i\underline{k} \cdot \underline{x}\} \, dA,$$

$$\tag{12}$$

$$\underline{k} = \omega\hat{k}/c.$$

Here $\hat{n}$ is the unit outward normal to ∂D.

In Reference 1, the function

$$v(\underline{x}, \omega, \underline{\xi}) = j_0(\omega R/c), \quad R = |\underline{x} - \underline{\xi}| \tag{13}$$

was used. It was this function which led to an ill-conditioned integral equation for f. It is the authors' present point of view that this is to be rejected as a poor choice for $v(\underline{x}, \omega)$.

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References

[1] N. Bleistein and J. K. Cohen, Non-uniqueness in the inverse source problem in acoustics and electromagnetics, J. Math. Physics, 18, 2, 194-201 (1977).

[2] J. D. Jackson, *Classical Electrodynamics*, 2nd Ed., Wiley, New York (1975).